

Time variation of particle and anti-particle asymmetry in an expanding universe

Ryuichi Hotta, Takuya Morozumi

Graduate School of Science, Hiroshima University Higashi-Hiroshima, 739-8526, Japan

Hiroyuki Takata

Tomsk State Pedagogical University, Tomsk, 634061, Russia

Abstract

Particle number violating interactions wash out the primordial asymmetry of particle number density generated by some interaction satisfying Sakharov conditions for baryogenesis. In this paper, we study how the primordial asymmetry evolves in time under the presence of particle number violating interactions and in the environment of expanding universe. We introduce a complex scalar model with particle number violating mass terms and calculate the time evolution of the particle number density with non-equilibrium quantum field theory. We show how the time evolution of the number density depends on parameters, including the chemical potential related with the particle number, temperature, size of the particle number violating mass terms, and the expansion rate of the universe. Depending upon whether the chemical potential is larger or smaller than the rest mass of the scalar particle, behaviors of the number density are very different to each other. When the chemical potential is smaller than the mass, the interference among the contribution of oscillators with various momenta reduces the number density in addition to the dilution due to the expansion of universe. In opposite case, the oscillation of the particle number density lasts for a long time and the cancellation due to the interference does not occur.

I. INTRODUCTION

Exploring the origin of the matter and anti-matter asymmetry of our universe, its production mechanism and time evolution are very important issues. In many scenarios of baryogenesis [1] [2] and leptogenesis [3], baryon number (B) and lepton number (L) interactions are required so that the primordial asymmetry of the particle number can be generated. After it is generated, the particle number violating interactions must be frozen. Otherwise, the primordial asymmetry created will be washed out. In this regards, in the context of leptogenesis [3], there are studies of the effect $\Delta L = 2$ operator of the mass dimension 5 on the primordial $\frac{B}{3} - L_i$ ($i = e, \mu, \tau$) asymmetries. If the coefficient of the operator is too large, the primordial asymmetries will be completely washed out. Since the same operator generates the Majorana mass matrix for light neutrinos at low energy, constraints on its elements are obtained from the condition that a leptogenesis scenario succeeds. One can also argue whether they are compatible with neutrino masses, lepton mixing matrix, and the experimental limit on the neutrino-less double beta decay rate [4].

We introduce a scalar model with particle number violating mass terms to investigate the time dependence of the particle number density in expanding universe, where the scale factor has arbitrary time dependence. In numerical study, we focus on the case that the scale factor grows exponentially and study how a given initial particle number asymmetry evolves under the influence of the particle number violating mass terms and the expansion. The scalar field is written in terms of a complex Klein Gordon field and one can identify the time component of U(1) current as a particle number density. Baryogenesis with a complex field has been discussed in several literatures [5], [6], [7], where the time derivative of the phase of the scalar is identified with the baryon number density.

We adopt the non-equilibrium field theory which has been developed in the literatures [8],[9], [10],[11],[12], [13] so that one can study the time evolution of the expectation value of the particle number density. We employ functional method since one can naturally extend the present study so that interactions and condensates are incorporated. The present work, therefore, serves as a starting point when we include interactions besides the terms quadratic with respect to the field. Despite of no interactions beyond quadratic terms, under the environment of the expanding universe, the initial condition with non-zero asymmetry and the particle number violating mass terms lead to non-trivial time evolution of the particle and

anti-particle asymmetry. In the expectation value, the weight of each state is specified by a density matrix. The density matrix is written with the grand canonical form and it is specified with temperature and chemical potential. The functional form for the density matrix with non-zero chemical potential is constructed explicitly. In our study, the primordial asymmetry of the particle number density is given by choosing the value and the sign of the chemical potential.

We derive formulae for the expectation value of the particle number density in an analytic form. For exponentially expanding universe, the formulae are written with Hankel functions. The various limiting cases, e.g., the case of the vanishing and/or small expansion rate and the case for vanishing particle number violating mass, etc., can be easily obtained. In numerical study, one can change the coefficient of the particle number violating mass term and the expansion rate of the universe. One can also change initial conditions by specifying the temperature and the chemical potential in the density matrix. Therefore, in a unified way, one can study its time evolutions for the cases with different sets of parameters.

The paper is organized as follows. In section II, Lagrangian for the scalar model is given. The initial density matrix is also specified. In section III, using a two particle irreducible effective action, we solve the Schwinger Dyson equation for Green functions and obtain the particle number density at arbitrary time. In section IV, we present the numerical results and section V is devoted to summary. In appendix A, the derivation of particle number density for small expansion rate is given and in appendix B, that for the vanishing limit of the particle number violating mass term is obtained.

II. THE COMPLEX SCALAR MODEL WITH U(1) BREAKING

We start with a complex scalar model including a soft U(1) symmetry breaking mass term. The time component of the U(1) current is a particle number density,

$$\begin{aligned}
S &= \int d^4x \sqrt{-g} \mathcal{L}, \\
\mathcal{L} &= g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi + \frac{B^2}{2} (\phi^2 + \phi^{*2}) \\
&\quad - m_\phi^2 |\phi|^2 + \left(\frac{\alpha_2}{2} \phi^2 + h.c. \right) R + \alpha_3 |\phi|^2 R.
\end{aligned} \tag{1}$$

U(1) breaking terms are denoted with their coefficients ; B and α_2 . The metric $g_{\mu\nu}$ is given by that of Friedmann Robertson Walker,

$$g_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2). \quad (2)$$

The Riemann curvature is given as $R = 12H^2$ with $H = \frac{\dot{a}}{a}$. When α_2 and B are real parameters, the mass eigenstates of the scalar are the real part and the imaginary part of the complex scalar ϕ . By decomposing it into a real part ϕ_1 and an imaginary part ϕ_2 as $\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$, their masses are given as follows,

$$\begin{aligned} \tilde{m}_1^2(x^0) &= m_\phi^2 - B^2 - 12\alpha_3 H^2 - 12\alpha_2 H^2, \\ \tilde{m}_2^2(x^0) &= m_\phi^2 + B^2 - 12\alpha_3 H^2 + 12\alpha_2 H^2. \end{aligned} \quad (3)$$

The current associated with U(1) transformation $\phi' = \phi e^{i\theta}$ is [5],

$$\begin{aligned} j_\mu &= i(\phi^\dagger \partial_\mu \phi - \partial_\mu \phi^\dagger \phi), \\ &= \phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1. \end{aligned} \quad (4)$$

Next we study the density matrix which specifies the initial state. Since we have non-vanishing primordial asymmetry of the particle number density, the statistical density matrix has the following form with non-zero chemical potential,

$$\rho = \frac{e^{-\beta(H_0 - \mu N)}}{\text{tr} e^{-\beta(H_0 - \mu N)}}, \quad (5)$$

where H_0 corresponds to the Hamiltonian obtained by taking the U(1) breaking terms and curvature dependent terms turned off and N is a particle number operator defined as follows,

$$N = \int d^3x \sqrt{-g} j^0. \quad (6)$$

The expectation value of the U(1) current is written with the density matrix in Eq.(5).

$$\langle j_\mu(X) \rangle = \text{tr}(j_\mu(X) \rho). \quad (7)$$

In section III, we compute the expectation value with the Green function of 2 PI (particle irreducible) formalism. From the definition of the U(1) current in Eq.(4), the expectation value defined in Eq.(7) can be written in terms of the Green function,

$$G_{12}^{12}(x, y) \equiv \text{tr}(\phi_2(y) \phi_1(x) \rho). \quad (8)$$

The resulting formulae for the expectation value of the current is given as follows,

$$\langle j_\mu(X) \rangle = \left(\frac{\partial}{\partial x^\mu} - \frac{\partial}{\partial y^\mu} \right) G_{12}^{12}(x, y) \Big|_{x=y=X}. \quad (9)$$

III. SCHWINGER DYSON EQUATION FROM 2 PARTICLE IRREDUCIBLE EFFECTIVE ACTION

In this section, we derive 2 PI effective action and obtain the Schwinger Dyson equations for Green functions. By solving the Schwinger Dyson equations, we obtain an analytic form for the expectation value of the current for the case that the scale factor of the universe of arbitrary time dependence.

2 PI effective action in curved space time for O(N) theory is derived in [12] and the method employed can be also applied to the present model. In 2 PI formalism, one first introduces non-local source term denoted by K in addition to the usual local source term J .

$$e^{iW[J,K]} = \int d\phi e^{i[S + i \int \sqrt{-g(x)} d^4x c^{ab} J_i^a \phi_i^b + \frac{1}{2} \int d^4x d^4y \sqrt{-g(x)} c^{ab} c^{cd} \phi_i^b(x) K_{ij}^{ac}(x,y) \phi_j^d(y) \sqrt{-g(y)}]}. \quad (10)$$

where c^{ab} is the metric of in-in formalism [13] and $c^{11} = -c^{22} = 1$ and $c^{12} = c^{21} = 0$.

The Legendre transformation of W leads to the 2 PI effective action, which is a functional of Green function.

$$\Gamma[G, \hat{\phi}, g] = S[\hat{\phi}, g] + \frac{i}{2} \text{Tr} \text{Ln} G^{-1} + \frac{i}{2} \int d^4x \int d^4y M_{ij}^{ab}(x, y) G_{ij}^{ab}(y, x), \quad (11)$$

where S and M_{ij}^{ab} are given by,

$$S[\hat{\phi}, g] = \frac{1}{2} \int d^4x \sqrt{-g(x)} c^{ab} (g^{\mu\nu} \nabla_\mu \hat{\phi}_i^a \nabla_\nu \hat{\phi}_i^b - \tilde{m}_i^2 \hat{\phi}_i^a \hat{\phi}_i^b), \quad (12)$$

$$iM_{ij}^{ab}(x, y) = -c^{ab} \delta_{ij} \sqrt{-g(x)} (\nabla_x^\mu \nabla_\mu^x + \tilde{m}_j^2) \delta^4(x - y). \quad (13)$$

The variation of the 2 PI effective action with respect to the scalar field $\hat{\phi}$ leads to,

$$\frac{\delta \Gamma}{\delta \hat{\phi}_i^a(x)} = -\sqrt{-g(x)} c^{ab} \{ J_i^b(x) + c^{cd} \int d^4z \sqrt{-g(z)} K_{il}^{bc}(x, z) \hat{\phi}_l^d(z) \}, \quad (14)$$

and one obtains the following equation of motion for the scalar field $\hat{\phi}_i$.

$$c^{ab} (g^{\mu\nu} \nabla_\mu \nabla_\nu + \tilde{m}_i^2) \hat{\phi}_i^b = c^{ab} \{ J_i^b(x) + c^{cd} \int d^4z \sqrt{-g(z)} K_{il}^{bc}(x, z) \hat{\phi}_l^d(z) \}. \quad (15)$$

When the single source term J vanishes, the equation of motion for $\hat{\phi}$ is homogeneous and linear with respect to $\hat{\phi}$. Therefore $\hat{\phi} = 0$ is a solution in this case. The variation of the 2 PI effective action with respect to Green function G is the source term K ,

$$\frac{\delta \Gamma}{\delta G_{ij}^{ab}(x, y)} = -\frac{1}{2} c^{ac} c^{bd} \sqrt{-g(x)} K^{cd}(x, y)_{ij} \sqrt{-g(y)}. \quad (16)$$

Eq.(16) leads to two differential equations,

$$\begin{aligned}
(\nabla_x^\mu \nabla_\mu^x + \tilde{m}_m^2) G_{mn}^{ab}(x, y) &= -i \frac{1}{\sqrt{-g(x)}} c^{ab} \delta_{mn} \delta(x - y) \\
&+ \int d^4 z K_{ml}^{ac}(x, z) \sqrt{-g(z)} c^{cd} G_{ln}^{db}(z, y), \\
(\nabla_y^\mu \nabla_\mu^y + \tilde{m}_n^2) G_{mn}^{ab}(x, y) &= -i c^{ab} \delta_{mn} \delta(x - y) \frac{1}{\sqrt{-g(y)}} \\
&+ \int d^4 z G_{ml}^{ac}(x, z) c^{cd} \sqrt{-g(z)} K_{ln}^{db}(z, y). \tag{17}
\end{aligned}$$

The non-local source term K is related to the functional representation of the initial density matrix ρ introduced in Eq.(5) [13],

$$\langle \phi^1 | \rho | \phi^2 \rangle = C \exp \left[\frac{i}{2} \int \int d^4 x d^4 y \sqrt{-g(x)} c^{ab} \phi_i^b(x) K_{ij}^{ac}(x, y) c^{cd} \phi_j^d(y) \sqrt{-g(y)} \right], \tag{18}$$

where C is a normalization factor and is determined so that the density matrix is normalized as $\text{tr} \rho = 1$. K is non-zero only if both x^0 and y^0 are the initial time. The resulting K has the following form,

$$K_{ij}^{ab}(x, y) = -i \delta(x_0) \delta(y_0) \kappa_{ij}^{ab}(\mathbf{x} - \mathbf{y}), \tag{19}$$

where κ specifies the space dependent part. Since it is invariant under translation, one can carry out the Fourier transformation on it.

$$\kappa(\mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \kappa(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \tag{20}$$

Let us derive the functional representation for the density matrix of Eq.(5) and determine κ .

$$\langle \phi^1 | \exp(-\beta(H_0 - \mu N)) | \phi^2 \rangle = \exp(\beta \mu \hat{N}) \langle \phi^1 | \exp(-\beta H_0) | \phi^2 \rangle \tag{21}$$

Note that ϕ^a ($a = 1, 2$) represents two components scalars.

$$\phi^a = \begin{pmatrix} \phi_1^a \\ \phi_2^a \end{pmatrix}. \tag{22}$$

We assume that the particle number violating term B^2 turned on when the universe begins to expand at $x^0 = 0$. The initial value for the scale factor is a_0 . Since the Hamiltonian H_0 and the particle number N commute with each other, the exponential factors in the

grand canonical distribution function are factorized as shown in Eq.(21). \hat{N} is a functional derivative acting on ϕ^1 and corresponds to the number operator in Eq.(6).

$$\hat{N} = \int d^3x a_0^3 j^0 = -i \int d^3x \left(\phi_2^1 \frac{\delta}{\delta \phi_1^1} - \phi_1^1 \frac{\delta}{\delta \phi_2^1} \right). \quad (23)$$

We first investigate the functional representation for the density matrix with zero chemical potential.

$$\begin{aligned} \langle \phi^1 | \exp(-\beta H_0) | \phi^2 \rangle &= \int_{\phi(u=\beta)=\phi^1, \phi(u=0)=\phi^2} d\phi \exp(-S_E) \\ &= C_0 \exp(-S_{Ecl}^{\mu=0}[\phi^1, \phi^2]). \end{aligned} \quad (24)$$

where S_E is a Euclidean action for the complex scalar field and $S_{Ecl}^{\mu=0}$ is the one for the classical trajectory with the boundary conditions at the Euclidean time $u = 0$ and $u = \beta$. C_0 is a constant. Explicitly S_E is given as,

$$S_E = \int_0^\beta du \int d^3x a_0^3 \left[\frac{\partial \phi^\dagger}{\partial u} \frac{\partial \phi}{\partial u} + \frac{\nabla \phi^\dagger \cdot \nabla \phi}{a_0^2} + m_\phi^2 \phi^\dagger \phi \right]. \quad (25)$$

and $S_{Ecl}^{\mu=0}$ becomes,

$$S_{Ecl}^{\mu=0}[\phi^1, \phi^2] = -\frac{a_0^6}{2} \sum_{i,j=1,2} \int \frac{d^3k}{(2\pi)^3} \phi_i^b(\mathbf{k}) c^{ab} c^{cd} \kappa_{ij}^{0ac}(-\mathbf{k}) \phi_j^d(-\mathbf{k}). \quad (26)$$

κ^0 represents κ defined in Eq.(20) for the zero chemical potential case. One can find,

$$\begin{aligned} \kappa_{ij}^{011}(-\mathbf{k}) &= \kappa_{ij}^{022}(-\mathbf{k}) = -\frac{1}{a_0^3} \frac{\omega(k) \cosh \beta \omega(k)}{\sinh \beta \omega(k)} \delta_{ij}, \\ \kappa_{ij}^{012}(-\mathbf{k}) &= \kappa_{ij}^{021}(-\mathbf{k}) = -\frac{1}{a_0^3} \frac{\omega(k)}{\sinh \beta \omega(k)} \delta_{ij}, \end{aligned} \quad (27)$$

where $\omega(k) = \sqrt{\frac{k^2}{a_0^2} + m_\phi^2}$. To obtain the functional representation of the density matrix for non-zero chemical potential, one notes the action of $\exp(\mu \beta \hat{N})$ generates $O(2)$ rotation among ϕ_1, ϕ_2 with a complex angle $i\mu\beta$,

$$\exp(\mu \beta \hat{N}) \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix} = O(i\mu\beta) \begin{pmatrix} \phi_1^1 \\ \phi_2^1 \end{pmatrix}, \quad (28)$$

where $O(i\mu\beta)$ is a rotation matrix,

$$O(i\mu\beta) = \begin{pmatrix} \cosh \mu\beta & -i \sinh \mu\beta \\ +i \sinh \mu\beta & \cosh \mu\beta \end{pmatrix}. \quad (29)$$

Therefore the action of $\exp(\mu\beta\hat{N})$ replaces ϕ^1 with $O(i\mu\beta)\phi^1$. The resulting functional representation of the density matrix for non-zero chemical potential is,

$$\begin{aligned} \langle \phi^1 | \exp(-\beta(H_0 - \mu N)) | \phi^2 \rangle &= \langle O(i\mu\beta)\phi^1 | \exp(-\beta H_0) | \phi^2 \rangle \\ &\equiv C \exp(-S_{cl}^\mu[\phi^1, \phi^2]), \end{aligned} \quad (30)$$

where,

$$\begin{aligned} S_{cl}^\mu[\phi^1, \phi^2] &= S_{cl}^{\mu=0}[O(i\mu\beta)\phi^1, \phi^2] \\ &= -\frac{a_0^6}{2} \sum_{i,j=1,2} \int \frac{d^3k}{(2\pi)^3} \phi_i^b(\mathbf{k}) c^{ab} c^{cd} \kappa_{ij}^{ac}(-\mathbf{k}) \phi_j^d(-\mathbf{k}). \end{aligned} \quad (31)$$

κ for non-zero chemical potential is given as,

$$\begin{aligned} \kappa_{ij}^{11}(-\mathbf{k}) &= \kappa_{ij}^{22}(-\mathbf{k}) = -\frac{1}{a_0^3} \frac{\omega(k) \cosh \beta\omega(k)}{\sinh \beta\omega(k)} \delta_{ij}, \\ \kappa_{ij}^{12}(-\mathbf{k}) &= -\frac{1}{a_0^3} \frac{\omega(k)}{\sinh \beta\omega(k)} O_{ij}^T(i\mu\beta), \\ \kappa_{ij}^{21}(-\mathbf{k}) &= -\frac{1}{a_0^3} \frac{\omega(k)}{\sinh \beta\omega(k)} O_{ij}(i\mu\beta). \end{aligned} \quad (32)$$

The normalization factor C can be determined by the condition $\text{Tr}(\rho) = 1$.

$$\langle \phi^1 | \rho | \phi^2 \rangle = \frac{\exp(-S_{cl}^\mu[\phi^1, \phi^2])}{\int d\phi_1 d\phi_2 \exp[-S_{cl}^\mu[\phi, \phi]]}, \quad (33)$$

where,

$$\begin{aligned} S_{cl}^\mu[\phi, \phi] &= a_0^3 \int \frac{d^3k}{(2\pi)^3} \frac{\omega(\mathbf{k})(\cosh \beta\omega(\mathbf{k}) - \cosh \beta\mu)}{\sinh \beta\omega(k)} \phi_i(\mathbf{k}) \phi_i(-\mathbf{k}) \\ &= \frac{1}{2} \int d^3\mathbf{x} d^3\mathbf{y} \phi_i(\mathbf{x}) D(\mathbf{x} - \mathbf{y}) \phi_i(\mathbf{y}), \end{aligned} \quad (34)$$

with $D(\mathbf{r})$ defined as,

$$D(\mathbf{r}) = 2a_0^3 \int \frac{d^3k}{(2\pi)^3} \frac{\omega(\mathbf{k})(\cosh \beta\omega(\mathbf{k}) - \cosh \beta\mu)}{\sinh \beta\omega(k)} \exp(-i\mathbf{r} \cdot \mathbf{k}). \quad (35)$$

The functional representation of the density matrix in Eq.(33) is used for obtaining the initial condition of the Green functions which are needed to solve the differential equations of Eq.(17). The Green function at $x^0 = y^0 = 0$ is defined as,

$$\begin{aligned} G_{ij}^{ab}(\mathbf{x}, x^0 = 0, \mathbf{y}, y^0 = 0) &= \text{Tr}[\hat{\phi}_j(\mathbf{y}) \hat{\phi}_i(\mathbf{x}) \rho], \\ &= \frac{\int d\phi_1 d\phi_2 \phi_j(\mathbf{y}) \phi_i(\mathbf{x}) \exp[-S_{cl}^\mu[\phi, \phi]]}{\int d\phi_1 d\phi_2 \exp[-S_{cl}^\mu[\phi, \phi]]}, \end{aligned} \quad (36)$$

and it can be computed with the generating functional,

$$\begin{aligned} W[J] &= \frac{\int d\phi_1 d\phi_2 \exp[-S_{cl}^\mu[\phi, \phi] + \int d^3\mathbf{x} J_i(\mathbf{x}) \phi_i(\mathbf{x})]}{\int d\phi_1 d\phi_2 \exp[-S_{cl}^\mu[\phi, \phi]]}, \\ &= \exp\left[\frac{1}{2} \int d^3x d^3y J_i(x) D^{-1}(x-y) J_i(y)\right]. \end{aligned} \quad (37)$$

Differentiating $W[J]$ with the source term twice, one obtains,

$$G_{ij}^{ab}(\mathbf{x}, x^0 = 0, \mathbf{y}, y^0 = 0) = \frac{\delta^2 W[J]}{\delta J_i(\mathbf{x}) \delta J_j(\mathbf{y})} \Big|_{J=0} = D^{-1}(\mathbf{x} - \mathbf{y}) \delta_{ij}, \quad (38)$$

where $D^{-1}(\mathbf{x} - \mathbf{y})$ satisfies

$$\int d^3\mathbf{y} D(\mathbf{x} - \mathbf{y}) D^{-1}(\mathbf{y} - \mathbf{z}) = \delta^3(\mathbf{x} - \mathbf{z}). \quad (39)$$

The Fourier transformation of $D(\mathbf{x} - \mathbf{y})$ and its inverse $D^{-1}(\mathbf{x} - \mathbf{y})$ are,

$$\begin{aligned} D(\mathbf{k}) &= 2a_0^3 \omega(k) \frac{\cosh \beta \omega(k) - \cosh \beta \mu}{\sinh \beta \omega(k)}, \\ D^{-1}(\mathbf{k}) &= \frac{1}{2a_0^3 \omega(k)} \left[\frac{\sinh \beta \omega(k)}{\cosh \beta \omega(k) - \cosh \beta \mu} \right]. \end{aligned} \quad (40)$$

Next we define the Fourier transform of the Green functions,

$$G_{ij}^{ab}(x, y) = \int \frac{d^3k}{(2\pi)^3} G_{ij}^{ab}(x^0, y^0, \mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (41)$$

Using Eq.(39) and Eq.(40), we obtain the initial value of the Fourier transformation of the Green function,

$$\begin{aligned} G_{ij}^{ab}(x^0 = 0, y^0 = 0, \mathbf{k}) &= \delta_{ij} \frac{1}{D(\mathbf{k})}, \\ &= \delta_{ij} \frac{1}{2\omega(\mathbf{k}) a_0^3} \left[\frac{\sinh \beta \omega(\mathbf{k})}{\cosh \beta \omega(\mathbf{k}) - \cosh \beta \mu} \right]. \end{aligned} \quad (42)$$

Since we obtain the initial condition of Green function, one can use it to solve the Schwinger Dyson equations.

In Friedman Robertson Walker metric, the Laplacian is given as,

$$\nabla_\mu \nabla^\mu = \frac{\partial^2}{\partial x^{02}} - \frac{1}{a(x^0)^2} \nabla \cdot \nabla + 3 \frac{\dot{a}}{a} \frac{\partial}{\partial x^0} \quad (43)$$

Therefore, the Fourier transformation of Green functions satisfy,

$$\begin{aligned}
& \left(\frac{\partial^2}{\partial x^{02}} + \frac{\mathbf{k}^2}{a(x^0)^2} + \tilde{m}_m(x^0)^2 + 3H \frac{\partial}{\partial x^0} \right) G_{mn}^{ab}(x^0, y^0, \mathbf{k}) \\
&= -i \frac{c^{ab}}{a(x^0)^3} \delta(x^0 - y^0) \delta_{mn} - i \delta(x^0) a_0^3 \kappa_{ml}^{ac}(\mathbf{k}) c^{cd} G_{ln}^{db}(0, y^0, \mathbf{k}), \\
& \left(\frac{\partial^2}{\partial y^{02}} + \frac{\mathbf{k}^2}{a(y^0)^2} + \tilde{m}_n(y^0)^2 + 3H \frac{\partial}{\partial y^0} \right) G_{mn}^{ab}(x^0, y^0, \mathbf{k}) \\
&= -i \frac{c^{ab}}{a(y^0)^3} \delta(x^0 - y^0) \delta_{mn} - i \delta(y^0) a_0^3 G_{ml}^{ac}(x^0, 0, \mathbf{k}) c^{cd} \kappa_{ln}^{db}(\mathbf{k}),
\end{aligned} \tag{44}$$

To solve Eq.(44), we introduce \hat{G} through the following equation.

$$G_{mn}^{ab}(x^0, y^0, k) = \left(\frac{a_0}{a(x^0)} \frac{a_0}{a(y^0)} \right)^{\frac{3}{2}} \hat{G}_{mn}^{ab}(x^0, y^0, k). \tag{45}$$

The differential equations are rewritten as,

$$\begin{aligned}
& \left(\frac{\partial^2}{\partial x^{02}} + \frac{\mathbf{k}^2}{a(x^0)^2} + \overline{m}_m(x^0)^2 \right) \hat{G}_{mn}^{ab}(x^0, y^0, \mathbf{k}) \\
&= -i \frac{c^{ab}}{a_0^3} \delta(x^0 - y^0) \delta_{mn} - i \delta(x^0) a_0^3 \kappa_{ml}^{ac}(\mathbf{k}) c^{cd} \hat{G}_{ln}^{db}(0, y^0, \mathbf{k}),
\end{aligned} \tag{46}$$

$$\begin{aligned}
& \left(\frac{\partial^2}{\partial y^{02}} + \frac{\mathbf{k}^2}{a(y^0)^2} + \overline{m}_n(y^0)^2 \right) \hat{G}_{mn}^{ab}(x^0, y^0, \mathbf{k}) \\
&= -i \frac{c^{ab}}{a_0^3} \delta(x^0 - y^0) \delta_{mn} - i \delta(y^0) a_0^3 \hat{G}_{ml}^{ac}(x^0, 0, \mathbf{k}) c^{cd} \kappa_{ln}^{db}(\mathbf{k}),
\end{aligned} \tag{47}$$

where $\overline{m}_m(x^0)^2 = \tilde{m}_m(x^0)^2 - \frac{9H^2}{4} - \frac{3}{2} \frac{dH}{dx^0}$. In the following, we denote two independent solutions of the homogeneous differential equation of Eq.(46) as $f_m(x^0)$ and $g_m(x^0)$.

$$\left(\frac{\partial^2}{\partial x^{02}} + \frac{\mathbf{k}^2}{a(x^0)^2} + \overline{m}_m(x^0)^2 \right) \begin{cases} f_m(x^0) = 0, \\ g_m(x^0) = 0. \end{cases} \tag{48}$$

To solve the differential equations for Green functions, we introduce the following four by four matrices.

$$\hat{G}(x^0, y^0, \mathbf{k}) = \begin{pmatrix} \hat{G}_{11}(x^0, y^0, \mathbf{k}) & \hat{G}_{12}(x^0, y^0, \mathbf{k}) \\ \hat{G}_{21}(x^0, y^0, \mathbf{k}) & \hat{G}_{22}(x^0, y^0, \mathbf{k}) \end{pmatrix}. \tag{49}$$

where each $\hat{G}_{ij}(x^0, y^0, \mathbf{k})$ is given by a two by two matrix.

$$\hat{G}_{ij}(x^0, y^0, \mathbf{k}) = \begin{pmatrix} \hat{G}_{ij}^{11}(x^0, y^0, \mathbf{k}) & \hat{G}_{ij}^{12}(x^0, y^0, \mathbf{k}) \\ \hat{G}_{ij}^{21}(x^0, y^0, \mathbf{k}) & \hat{G}_{ij}^{22}(x^0, y^0, \mathbf{k}) \end{pmatrix}. \tag{50}$$

In this notation, c and κ are given as,

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{11}(-\mathbf{k}) & \kappa_{12}(-\mathbf{k}) \\ \kappa_{21}(-\mathbf{k}) & \kappa_{22}(-\mathbf{k}) \end{pmatrix}. \quad (51)$$

where each $\kappa_{ij}(-\mathbf{k})$ is a two by two matrix and is given by,

$$\kappa_{ij}(-\mathbf{k}) = \begin{pmatrix} \kappa_{ij}^{11}(-\mathbf{k}) & \kappa_{ij}^{12}(-\mathbf{k}) \\ \kappa_{ij}^{21}(-\mathbf{k}) & \kappa_{ij}^{22}(-\mathbf{k}) \end{pmatrix}. \quad (52)$$

Now let us solve Eq.(46) and Eq.(47). When $x^0 > y^0$, one first writes $\hat{G}(x^0, y^0)$ in terms of $\hat{G}(x^0, 0)$ and $\left. \frac{\partial \hat{G}(x^0, y^0)}{\partial y^0} \right|_{y^0=0}$.

$$\hat{G}_{mn}^{ab}(x^0, y^0, \mathbf{k}) = \hat{G}_{mn}^{ab}(x^0, 0, \mathbf{k})\omega_n(y^0) + \left. \frac{\partial \hat{G}_{mn}^{ab}(x^0, y^0, \mathbf{k})}{\partial y^0} \right|_{y^0=0} z_n(y^0), \quad (53)$$

$$\left. \frac{\partial \hat{G}_{mn}^{ab}(x^0, y^0, \mathbf{k})}{\partial y^0} \right|_{y^0=0} = -ia_0^3 \hat{G}_{ml}^{ac}(x^0, 0, \mathbf{k}) c^{cd} \kappa_{ln}^{db}(-\mathbf{k}), \quad (54)$$

Next we write $G(x^0, 0)$ with $G(0, 0)$ as,

$$\hat{G}_{mn}^{ab}(x^0, 0, \mathbf{k}) = \omega_m(x^0) \hat{G}_{mn}^{ab}(0, 0, \mathbf{k}) + z_m(x^0) \left. \frac{\partial \hat{G}_{mn}^{ab}(x^0, 0, \mathbf{k})}{\partial x^0} \right|_{x^0=0}, \quad (55)$$

$$\left. \frac{\partial \hat{G}_{mn}^{ab}(x^0, 0, \mathbf{k})}{\partial x^0} \right|_{x^0=0} = -i \frac{c^{ab}}{a_0^3} \delta_{mn} - ia_0^3 \kappa_{ml}^{ac}(-\mathbf{k}) c^{cd} G_{ln}^{db}(0, 0, \mathbf{k}), \quad (56)$$

where $w_n(x^0)$ and $z_n(x^0)$ are defined as,

$$\begin{aligned} w_n(x^0) &= \frac{f_n(x^0)\dot{g}_n(0) - g_n(x^0)\dot{f}_n(0)}{f_n(0)\dot{g}_n(0) - g_n(0)\dot{f}_n(0)}, \\ z_n(x^0) &= \frac{-f_n(x^0)g_n(0) + g_n(x^0)f_n(0)}{f_n(0)\dot{g}_n(0) - g_n(0)\dot{f}_n(0)}. \end{aligned} \quad (57)$$

Using Eqs.(53-56), one can write $\hat{G}(x^0, y^0)$ in terms of $\hat{G}(0, 0)$ where $\hat{G}(0, 0)$ is obtained in Eq.(42) in the previous section. To compute all components of \hat{G} , one introduces the

diagonal matrices $w(x^0)$ and $z(x^0)$,

$$w(x^0) = \begin{pmatrix} w_1(x^0) & 0 & 0 & 0 \\ 0 & w_1(x^0) & 0 & 0 \\ 0 & 0 & w_2(x^0) & 0 \\ 0 & 0 & 0 & w_2(x^0) \end{pmatrix}, \quad (58)$$

$$z(x^0) = \begin{pmatrix} z_1(x^0) & 0 & 0 & 0 \\ 0 & z_1(x^0) & 0 & 0 \\ 0 & 0 & z_2(x^0) & 0 \\ 0 & 0 & 0 & z_2(x^0) \end{pmatrix}. \quad (59)$$

Using them, one can write the solution $\hat{G}(x^0, y^0)$ for $x^0 > y^0$ as,

$$\begin{aligned} \hat{G}(x^0, y^0, \mathbf{k}) &= (w(x^0) - z(x^0)ia_0^3\kappa c)G(0, 0, \mathbf{k})(w(y^0) - ic\kappa a_0^3z(y^0)) \\ &\quad - iz(x^0)\frac{c}{a_0^3}(w(y^0) - ic\kappa a_0^3z(y^0)). \end{aligned} \quad (60)$$

For $x^0 < y^0$, one can also write the solution in the matrix form similar to Eq.(60). The result is,

$$\begin{aligned} \hat{G}(x^0, y^0, \mathbf{k}) &= (w(x^0) - z(x^0)ia_0^3\kappa c)G(0, 0, \mathbf{k})(w(y^0) - ic\kappa a_0^3z(y^0)) \\ &\quad - i(w(x^0) - z(x^0)ia_0^3\kappa c)\frac{c}{a_0^3}z(y^0). \end{aligned} \quad (61)$$

By combining Eq.(60) with Eq.(61), one obtains,

$$\begin{aligned} G(x^0, y^0, \mathbf{k}) &= \left(\frac{a_0}{a(x^0)} \frac{a_0}{a(y^0)} \right)^{\frac{3}{2}} \left[(w(x^0) - z(x^0)ia_0^3\kappa c)G(0, 0, \mathbf{k})(w(y^0) - ic\kappa a_0^3z(y^0)) - z(x^0)\kappa z(y^0) \right. \\ &\quad \left. - i\theta(x^0 - y^0)z(x^0)\frac{c}{a_0^3}\omega(y^0) - i\theta(y^0 - x^0)\omega(x^0)\frac{c}{a_0^3}z(y^0) \right]. \end{aligned} \quad (62)$$

Now we are ready to write all the Green functions explicitly. The diagonal elements, \hat{G}_{ii} ($i = 1, 2$) are given as,

$$\begin{aligned} \hat{G}_{ii}(x^0, y^0, k) &= \frac{(w_i(x^0)w_i(y^0) + \omega(k)^2 z_i(x^0)z_i(y^0)) \sinh \beta\omega(k)}{2a_0^3\omega(k)(\cosh \beta\omega(k) - \cosh \beta\mu)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &\quad + \frac{i}{2a_0^3}(w_i(x^0)z_i(y^0) - z_i(x^0)w_i(y^0)) \begin{pmatrix} \epsilon(x^0 - y^0) & -1 \\ 1 & -\epsilon(y^0 - x^0) \end{pmatrix}, \end{aligned} \quad (63)$$

where $\epsilon(x^0 - y^0) = \theta(x^0 - y^0) - \theta(y^0 - x^0)$. The off-diagonal ones, $\hat{G}_{ij}(i \neq j)$ are given by,

$$\begin{aligned}\hat{G}_{12}(x^0, y^0, k) &= \frac{\sinh \beta\mu}{2a_0^3(\cosh \beta\omega(k) - \cosh \beta\mu)}(-z_2(y^0)w_1(x^0) + w_2(y^0)z_1(x^0)) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \\ \hat{G}_{21}(x^0, y^0, k) &= \frac{\sinh \beta\mu}{2a_0^3(\cosh \beta\omega(k) - \cosh \beta\mu)}(-z_2(x^0)w_1(y^0) + w_2(x^0)z_1(y^0)) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.\end{aligned}\tag{64}$$

We also write G_{12}^{12} explicitly,

$$G_{12}^{12}(x^0, y^0, \mathbf{k}) = \left(\frac{a_0}{a(x^0)} \frac{a_0}{a(y^0)} \right)^{\frac{3}{2}} \frac{\sinh \beta\mu}{2a_0^3(\cosh \beta\omega(k) - \cosh \beta\mu)}(-z_2(y^0)w_1(x^0) + w_2(y^0)z_1(x^0)).\tag{65}$$

Using the result, one can write the current density,

$$\begin{aligned}\langle j_0(x^0) \rangle &= \left(\frac{a_0}{a(x^0)} \right)^3 \int \frac{d^3k}{(2\pi)^3} \frac{\sinh \beta\mu}{2a_0^3(\cosh \beta\omega(k) - \cosh \beta\mu)} \\ &\quad [-\dot{w}_1(x^0)z_2(x^0) + w_1(x^0)\dot{z}_2(x^0) - \dot{w}_2(x^0)z_1(x^0) + w_2(x^0)\dot{z}_1(x^0)].\end{aligned}\tag{66}$$

Now let us examine the solutions of homogeneous differential equations of Eq.(48) for the case that the scale factor grows exponentially with respect to time,

$$a(x^0) = a_0 \exp(Hx^0).\tag{67}$$

The masses \tilde{m}_m in Eq.(3) are independent of time and an analytic form for the Green functions can be obtained. In this case one can introduce the conformal time [14],

$$\eta = -\frac{k}{Ha(x^0)} = -\frac{k}{He^{Hx^0}}.\tag{68}$$

where we can set $a(x_0 = 0) = a_0 = 1$ without loss of generality. One finds the f_m and g_m ($m = 1, 2$) satisfy the differential equation for Bessel function,

$$\left[\frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} + 1 + \frac{\rho_m^2}{\eta^2} \right] \begin{cases} f_m(\eta) = 0 \\ g_m(\eta) = 0, \end{cases}\tag{69}$$

where ρ_m ($m = 1, 2$) is given as,

$$\rho_m = \frac{\overline{m}_m}{H},\tag{70}$$

$$\begin{aligned}\rho_1 &= \sqrt{\frac{m_\phi^2}{H^2} - \frac{B^2}{H^2} - 12(\alpha_3 + \alpha_2) - \frac{9}{4}}, \\ \rho_2 &= \sqrt{\frac{m_\phi^2}{H^2} + \frac{B^2}{H^2} - 12(\alpha_3 - \alpha_2) - \frac{9}{4}}.\end{aligned}\tag{71}$$

One can choose the Hankel function $H_{i\rho_m}$ as one of the solution.

$$f_m(x^0) = H_{i\rho_m}[\eta] = \frac{1}{\sinh \rho_m \pi} \left(e^{\rho_m \pi} J_{i\rho_m}[\eta] - J_{-i\rho_m}[\eta] \right),\tag{72}$$

where we also show the formula which relates the Hankel function to the Bessel function. The current density of Eq.(66) is also written in terms of the derivative with conformal time. Using the relation of the derivatives,

$$\frac{\partial}{\partial x^0} = \frac{k}{a(x^0)} \frac{\partial}{\partial \eta},\tag{73}$$

one can write w_n and z_n in Eq.(57),

$$\begin{aligned}w_n(x^0) &= \frac{H_{i\rho_n}[\eta] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}^*[\eta] H_{i\rho_n}'[\eta_0]}{H_{i\rho_n}[\eta_0] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}^*[\eta_0] H_{i\rho_n}'[\eta_0]}, \\ z_n(x^0) &= \frac{-H_{i\rho_n}[\eta] H_{i\rho_n}^*[\eta_0] + H_{i\rho_n}^*[\eta] H_{i\rho_n}[\eta_0]}{k(H_{i\rho_n}[\eta_0] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}'[\eta_0] H_{i\rho_n}^*[\eta_0])},\end{aligned}\tag{74}$$

where $H' \equiv \frac{\partial H}{\partial \eta}$ and $\eta_0 = -\frac{k}{H}$. The time derivatives of w_n and z_n are also written with the derivatives with the conformal time.

$$\begin{aligned}\dot{w}_n(x^0) &= \frac{k}{a(x^0)} \frac{H_{i\rho_n}'[\eta] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}^*[\eta] H_{i\rho_n}'[\eta_0]}{H_{i\rho_n}[\eta_0] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}^*[\eta_0] H_{i\rho_n}'[\eta_0]}, \\ \dot{z}_n(x^0) &= \frac{1}{a(x^0)} \frac{-H_{i\rho_n}'[\eta] H_{i\rho_n}^*[\eta_0] + H_{i\rho_n}^*[\eta] H_{i\rho_n}[\eta_0]}{(H_{i\rho_n}[\eta_0] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}'[\eta_0] H_{i\rho_n}^*[\eta_0])}.\end{aligned}\tag{75}$$

Furthermore, we introduce W_n and Z_n as functions of the conformal time.

$$\begin{aligned}W_n[\eta, \eta_0] &= H_{i\rho_n}[\eta] H_{i\rho_n}^*[\eta_0] - H_{i\rho_n}^*[\eta] H_{i\rho_n}'[\eta_0], \\ Z_n[\eta, \eta_0] &= -H_{i\rho_n}[\eta] H_{i\rho_n}^*[\eta_0] + H_{i\rho_n}^*[\eta] H_{i\rho_n}[\eta_0].\end{aligned}\tag{76}$$

With these formulas, one can write the current density of Eq.(66) with the conformal time.

$$\begin{aligned}\langle j_0(x^0) \rangle &= e^{-4Hx^0} \int \frac{d^3 k}{(2\pi)^3} \frac{\sinh \beta \mu}{(\cosh \beta \omega(k) - \cosh \beta \mu)} \\ &\quad \left[\frac{-W_1'[\eta, \eta_0] Z_2[\eta, \eta_0] + W_1[\eta, \eta_0] Z_2'[\eta, \eta_0] - W_2'[\eta, \eta_0] Z_1[\eta, \eta_0] + W_2[\eta, \eta_0] Z_1'[\eta, \eta_0]}{2W_1[\eta_0, \eta_0] W_2[\eta_0, \eta_0]} \right].\end{aligned}\tag{77}$$

Below we investigate some extreme limit of the particle number density of Eq.(66) and Eq.(77). We first study the small limit of the Hubble parameter H of Eq.(77). When H is small, $\rho_i = \sqrt{\frac{m_i^2}{H^2}}$ and $x = \frac{k}{H}$ become large. In appendix A, we derive the approximate formula for the small H limit. From Eq.(A22), one obtains the current density for small H limit,

$$\begin{aligned} \langle j_0(x^0) \rangle = & e^{-3Hx^0} \int \frac{d^3k}{(2\pi)^3} \frac{\sinh \beta\mu}{(\cosh \beta\omega(k) - \cosh \beta\mu)} \\ & \frac{1}{2} \left[\left\{ \left(\frac{\omega_1\omega_1(x^0)}{\omega_2\omega_2(x^0)} \right)^{\frac{1}{2}} + \left(\frac{\omega_2\omega_2(x^0)}{\omega_1\omega_1(x^0)} \right)^{\frac{1}{2}} \right\} \sin x(f(a, \sigma_1) - f(1, \sigma_1)) \sin x(f(a, \sigma_2) - f(1, \sigma_2)) \right. \\ & \left. + \left\{ \left(\frac{\omega_1\omega_2(x^0)}{\omega_2\omega_1(x^0)} \right)^{\frac{1}{2}} + \left(\frac{\omega_2\omega_1(x^0)}{\omega_1\omega_2(x^0)} \right)^{\frac{1}{2}} \right\} \cos x(f(a, \sigma_1) - f(1, \sigma_1)) \cos x(f(a, \sigma_2) - f(1, \sigma_2)) \right], \end{aligned} \quad (78)$$

where $\sigma_i = \frac{m_i}{k}$, $\omega_i = \sqrt{k^2 + m_i^2}$ and $\omega_i(x^0) = \sqrt{\frac{k^2}{a(x^0)^2} + m_i^2}$. $f(a, \sigma) = \sqrt{\frac{1}{a^2} + \sigma^2} - \sigma \sinh^{-1}(\sigma a)$. One can also take the limit that the expansion rate H vanishes. (See Eq.(A26).)

$$\begin{aligned} \lim_{H \rightarrow 0} \langle j_0(x^0) \rangle = & \int \frac{d^3k}{(2\pi)^3} \frac{\sinh \beta\mu}{(\cosh \beta\omega(k) - \cosh \beta\mu)} \\ & \left[\cos(\omega_1 - \omega_2)x^0 + \frac{1}{2} \left\{ \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} - 2 \right\} \sin \omega_1 x^0 \sin \omega_2 x^0 \right]. \end{aligned} \quad (79)$$

Another interesting limit is the case that particle number violating mass term B and α_2 vanish. In this case, the particle number density per unit comoving volume is conserved. Since the comoving volume grows as $a(x^0)^3$, the density in a unit physical volume decreases as $a(x^0)^{-3}$. In appendix B, we explicitly derive the current density for the case with $B = \alpha_2 = 0$ and the result is given as follows,

$$\lim_{B \rightarrow 0, \alpha_2 \rightarrow 0} \langle j_0(x^0) \rangle = \left(\frac{a_0}{a(x^0)} \right)^3 \int \frac{d^3k}{(2\pi)^3} \frac{\sinh \beta\mu}{(\cosh \beta\omega(k) - \cosh \beta\mu)}. \quad (80)$$

IV. NUMERICAL RESULTS

So far we take various limits and derive the corresponding formulae. In this section, we study the exact formulae and the time dependence of the current density numerically. We focus on the case that the scale factor grows exponentially with respect to time. Since the

mode functions satisfy Eq.(69), the effective masses; \bar{m}_1 and \bar{m}_2 are given as follows,

$$\bar{m}_1^2 = m_\phi^2 - B^2 - 12(\alpha_3 + \frac{3}{16})H^2 - 12\alpha_2 H^2 \quad (81)$$

$$\bar{m}_2^2 = m_\phi^2 + B^2 - 12(\alpha_3 + \frac{3}{16})H^2 - 12\alpha_2 H^2. \quad (82)$$

When the expansion rate H is small and the condition on α_i ($i = 2, 3$),

$$\frac{|B|}{H} \gg \sqrt{|12\alpha_2|}, \quad \sqrt{|12\alpha_3 + \frac{3}{16}|} \quad (83)$$

is satisfied, the leading contribution to the effective masses is given by the one of the flat space-time case specified with B and m_ϕ . We study the case that the effective masses are the same as those in flat space-time case and the H dependent parts of Eq.(81) and Eq.(82) vanish.

$$\alpha_3 = \frac{-3}{16}, \quad \alpha_2 = 0. \quad (84)$$

Limiting ourselves to this case, ρ_m ($m = 1, 2$) in Eq.(71) can be determined by the coefficient of the particle number violating term B ,

$$\rho_1 = \frac{\sqrt{m_\phi^2 - B^2}}{H}, \quad \rho_2 = \frac{\sqrt{m_\phi^2 + B^2}}{H}. \quad (85)$$

Without loss of generality, the initial value of the scale factor a_0 can be set to unity. One can write the current density Eq.(77) as,

$$\langle j_0(X^0) \rangle = \int \frac{d^3k}{(2\pi)^3} h(\mathbf{k}, \mu, T) V(\eta, \eta_0, k), \quad (86)$$

where h denotes the momentum distribution for the current density and V denotes the time evolution factor. They are defined respectively as follows,

$$h(\mathbf{k}, \mu, T) = \frac{\sinh \beta\mu}{\cosh \beta\omega - \cosh \beta\mu}, \quad (87)$$

$$V(\eta, \eta_0, k) = e^{-4HX^0} \frac{-W'_1[\eta, \eta_0]Z_2[\eta, \eta_0] + W_1[\eta, \eta_0]Z'_2[\eta, \eta_0] + (1 \leftrightarrow 2)}{2W_1[\eta_0, \eta_0]W_2[\eta_0, \eta_0]}. \quad (88)$$

Because the time evolution factor V is unity at the initial time, one notes that the initial current density is simply given as,

$$\langle j_0(X^0 = 0) \rangle = \int \frac{d^3k}{(2\pi)^3} h(\mathbf{k}, \mu, T). \quad (89)$$

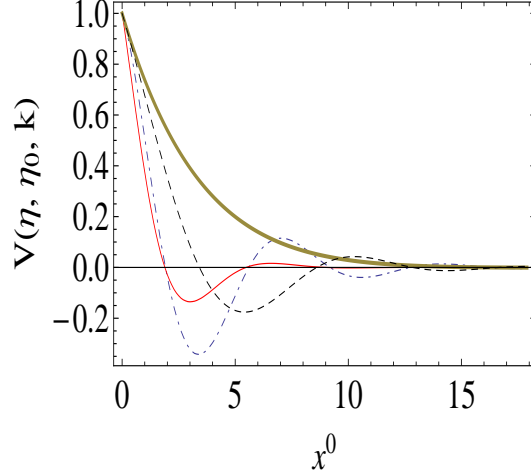


FIG. 1: The plot shows the time dependent factor $V(\eta, \eta_0, k)$ for different momentum k , the expansion rate H and particle number violating mass B . The chemical potential and the temperature are fixed as $(\mu, T) = (5, 20)$. The dash-dotted line shows the result for the case $(B, H, k) = (3, 0.1, 5)$, the solid line shows the result for the case $(3, 0.2, 5)$ and the thick solid line shows the result for the case $(1, 0.1, 5)$. The dashed line shows the result for the case with $(B, H, k) = (3, 0.1, 20)$.

In Fig.1, we show the time evolution factor $V(\eta, \eta_0, k)$. The period of oscillation tends to be long for the case that the mass squared difference $2B^2$ is small and the momentum k is large. The damping speed becomes faster as the expansion rate H is larger. In Figs. 2 and 3, we show the momentum distribution function $h(k, \mu, T)$ for cases with different values of chemical potential μ . In Fig. 2, the case for $m_\phi > \mu$ is shown with $m_\phi = 10$. Fig. 3 shows the opposite case, i.e., $m_\phi < \mu$. Behavior of the two cases is different to each other because for the latter case, $h(k, \mu, T)$ has a pole at the momentum satisfying $\mu = \omega(k)$. We also find that for very large momentum compared with the temperature T and the chemical potential μ , $h(k, \mu, T)$ becomes very small. Therefore one can set the upper limit of the momentum integration with a certain large momentum k_{\max} and one can carry out the momentum integration approximately.

Below, we carry out momentum integration and show the time variation of the current density. We set the upper limit of the momentum integration $k_{\max} = 200$. We show the parameter dependence for time evolution of the current density in Fig.4 ~ Fig.8. First, we show the dependence on the expansion rate H in Fig. 4 and Fig. 5. The expansion rate

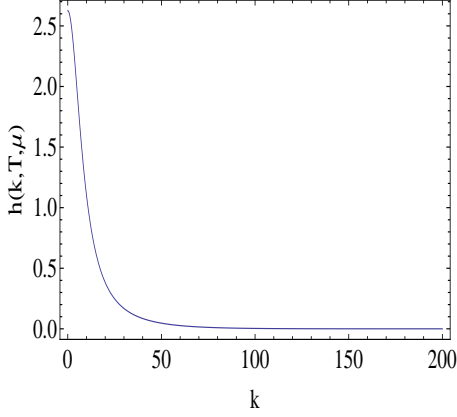


FIG. 2: Momentum dependence of the distribution function $h(k, \mu, T)$. We choose parameters $(\mu, T) = (5, 20)$.

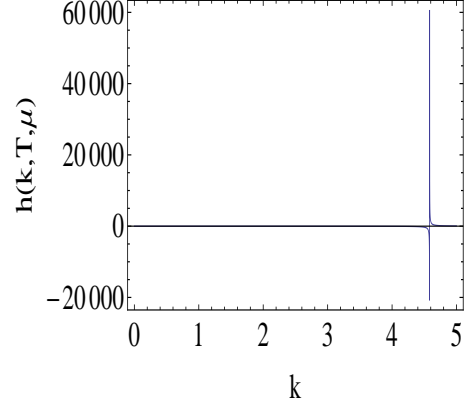


FIG. 3: Momentum dependence of the distribution function $h(k, \mu, T)$. We choose parameters $(\mu, T) = (11, 20)$.

affects the damping speed of the current density. In fact, as the expansion rate H becomes larger, the damping speed is faster. We notice that the current density is suppressed even for the case that the expansion rate H vanishes. This is clearly seen from the behavior of the thick solid line of Fig. 5. As shown in Fig. 1, the period of oscillation in V varies depending on momentum k . Therefore, contributions from different k interfere destructively and their sum becomes small. In Fig. 6, we plot B dependence for time evolution of the current density. The period of oscillation becomes shorter as the mass squared difference is larger.

In Fig. 7, we show the dependence on temperature T of the current density. It depends on the temperature only through the initial distribution function $h(k, \mu, T)$. As the temperature is higher, the initial current density becomes larger. We expect the oscillatory behavior will be more pronounced for low temperature case and Fig. 7 shows the tendency. When the temperature T is small compared with the mass scale m_ϕ , the oscillation period is determined by the inverse of mass difference $m_2 - m_1$. When temperature T is larger than the mass scale, the period will be proportional to $\frac{T}{m_\phi(m_2 - m_1)}$. Therefore, when the temperature T is larger than m_ϕ , the oscillation period becomes large. We also show μ dependence of the current density in Fig. 8. The chemical potential μ also influences the current density at the initial time. As the chemical potential becomes larger, the initial current density becomes larger.

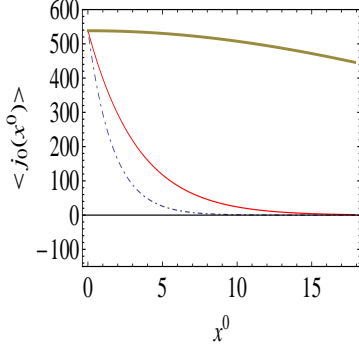


FIG. 4: Curvature (H) effect on time evolution of current density. The dash-dotted line, the solid line and the thick solid line show the case for $H = 0.2, 0.1$, and 0 respectively. $(B, \mu, T) = (1, 5, 20)$ for all the lines.

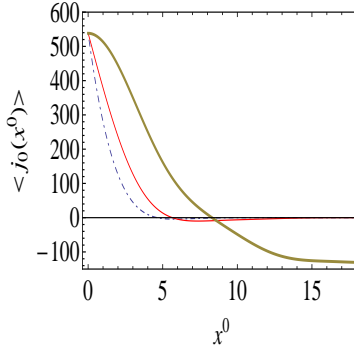


FIG. 5: Curvature (H) effect on time evolution of current density. The dash-dotted line, the solid line and the thick solid line show the case for $H = 0.2, 0.1$, and 0 respectively. $(B, \mu, T) = (3, 5, 20)$ for all the lines.

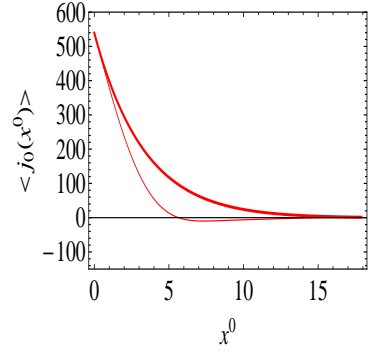


FIG. 6: The mass squared difference $(m_2^2 - m_1^2 = 2B^2)$ dependence of current density. The solid line and the thick solid line show the case $B = 3$ and $B = 1$ respectively. $(H, \mu, T) = (0.1, 5, 20)$ for all the lines.

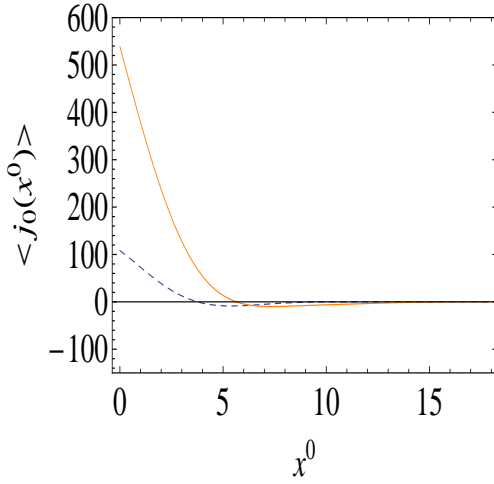


FIG. 7: Dependence on temperature T of time evolution of current density. The dashed line and the solid line show the case $T = 10$ and $T = 20$ respectively. $(B, H, \mu) = (3, 0.1, 5)$ for all the lines.

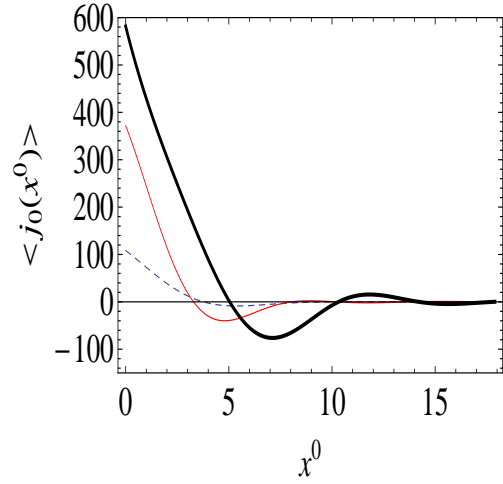


FIG. 8: Dependence on chemical potential μ of time evolution of current density. The dashed line, the solid line, the thick solid line show the case $\mu = 5, 11$ and 20 respectively. $(B, H, T) = (3, 0.1, 10)$ for all the lines.

The effect of the large chemical potential on the time dependence of the current density is very different from that of the small chemical potential. In Fig. 9, we pay attention to the damping speed and observe the distinctive behavior between the two cases, i.e., $\mu > m_\phi$ and $m_\phi < \mu$. We compare the time dependence of the current density normalized by their

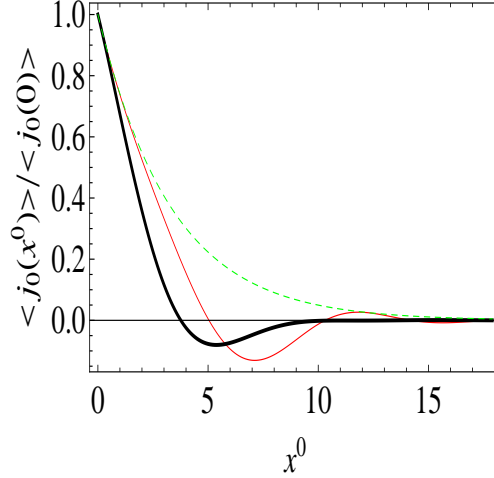


FIG. 9: We show the time dependence of the current densities normalized by their initial values. The case with $\mu > m_\phi$ and the case with $\mu < m_\phi$ are shown. The thin solid line shows the case with $\mu = 20$ and the thick solid line shows the case with $\mu = 5$, respectively. We choose $m_\phi = 10$ and $(B, H, T) = (3, 0.1, 10)$ for both cases. For comparison, with dashed line, we show the time dependence for the inverse of the universe's volume, i.e., e^{-3Hx^0} with $H = 0.1$.

initial values. When the chemical potential exceeds the mass scale m_ϕ (thin solid line) the oscillatory behavior lasts much longer than the case with the small chemical potential (thick solid line). The damping behavior $\sim e^{-3Hx^0}$ which is expected from the simple volume expansion of the universe is also shown with the dotted line. The exponential damping rate of the current density for the large chemical potential case ($\mu = 20$) accords with the one expected from the volume expansion. The damping effect due to the destructive interference can not be seen when the chemical potential is greater than m_ϕ . As shown in Fig.3, the momentum distribution has a pole at a certain momentum satisfying the condition $\mu = \omega(k)$. Therefore the absolute value of the distribution function h is very large within the small range of the momenta around the pole. From the behavior of the distribution function, one concludes that the contribution only from a certain momentum region is dominant for the case $m_\phi < \mu$ and the oscillation period of the current density is fixed even after integrating the distribution (h) \times time evolution factor (V) over all the momenta.

V. CONCLUSION AND DISCUSSION

We have studied how the primordial matter and anti-matter asymmetry in expanding universe evolves under the influence of the particle number violating interactions. To investigate its time variation, we have introduced the complex scalar field with the well-defined particle number density. The Lagrangian of the scalar field includes the mass term which breaks the particle number conservation. Such mass term can lead to the time evolution of the asymmetry existing at the beginning, however, it cannot produce the asymmetry itself. We have assumed that the particle number breaking mass term is turned on when the universe begins to expand. For the numerical calculation, we assume the universe expands exponentially with respect to time. Under the assumptions, the expectation value of the particle number density in later time is obtained and its formula has been given in an analytic form with the special function. The particle number density is written in terms of the momentum integration of the time dependent function $V(\eta, \eta_0, k)$ weighted with the distribution function (see Fig.1). By specifying the chemical potential and temperature in the density matrix, we have determined the initial condition for the particle number density. We have numerically calculated the evolution of the density and showed various cases with changing the expansion rate of the universe, the value of the particle number violating mass term, chemical potential and temperature. In particular, we have paid attention to the speed of decreasing of the particle number density. When the particle number is conserved, the density decreases in inversely proportional to the volume of the universe. When the particle number violating interaction is turned on, the behavior of the density is very different from that of the case without the interaction.

There are two typical cases. When the chemical potential is smaller than the mass of the complex scalar, besides the damping effect due to the expansion, the interference of contributions from various momenta also reduces the particle number density. On the other hand, when the chemical potential is larger than the scalar mass, the contribution from a certain momentum region is dominant. The resulting particle number density oscillates with a definite frequency in addition to the damping.

The phenomena of the single frequency dominance is related to the fact that the distribution function for the complex scalar boson has a pole at some momentum. In contrast to the case with small chemical potential, the interference does not occur and the density

continues to oscillate over the time until the density itself is suppressed by the expansion of the universe.

The phenomena of the decoherence with the interference and the coherence with large chemical potential will have some impact on concrete scenarios of matter and anti-matter problem. The decoherence effect has some impact on the dark matter problem. In the scenario of the asymmetric dark matter, the dark matter is a remnant of the matter and anti-matter annihilation. The matter and anti-matter oscillation, if it exists, can change the amount of the dark matter [15], [16]. If the decoherence occurs and the primordial asymmetry is washed out, the amount of the matter becomes nearly equal to the one of the anti-matter. The pair annihilation of matter and anti-matter leads to the further reduction of the dark matter.

Let us consider the case that the sign of the primordial particle number asymmetry is positive and is the same as that of the present asymmetry. Suppose the particle number violating mass term is so small that the sign of the asymmetry has been remained as positive. If this is the case, the strength of the particle number violating mass term will be determined with cosmological observation on the difference between the primordial asymmetry and the present one. Although we have assumed in the numerical calculation that universe expands exponentially with respect to time, it is also possible to extend to the case when the scale factor has more general dependence on time, such as a power law. In principle, one can reduce the problem to solving the linear differential with the scale factor. The current density can be written in terms of the solutions and the formulae similar to Eq.(77) will be obtained.

To extend the present model to a realistic one, we need to introduce new interactions and new degrees of freedom so that the primordial density can be generated.

Appendix A: Approximate formulae for the small H limit

In this appendix, we derive the approximate formulae of the particle number density, when the expansion rate H is small in Eq.(77). We start with the following integral representation for Hankel functions [17].

$$\omega_{\lambda}^R(-x) = -\frac{1}{\pi} \int_R d\zeta e^{ix \sin \zeta + i\lambda \zeta}, \quad (\text{A1})$$

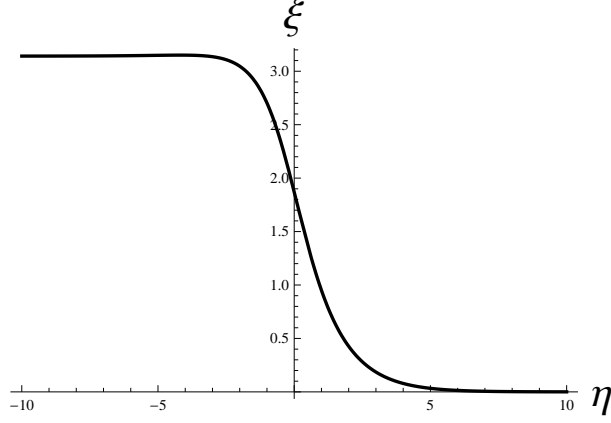


FIG. 10: The thick solid line shows the contour R .

where $\frac{k}{H} = x > 0$ with $\lambda = i\rho = i\frac{m}{H}$. We first derive the asymptotic form for the Hankel functions in the small H limit. When H is small, both x and λ are large. To obtain the approximate form, we can write,

$$\lambda = i\rho = i\sigma x, \quad \sigma = \frac{m}{k}. \quad (\text{A2})$$

Using the integral representation for the Hankel functions, we obtain the approximate form for them in large x limit. In the integral representation, R denotes the contour for the integration with respect to a complex variable $\zeta = \xi + i\eta$. The contour is shown in Fig. 10. The contour R is the curve which begins at $(\xi, \eta) = (\pi, -\infty)$ and ends at $(0, \infty)$. On the contour R , ξ varies within the range $[0, \pi]$. One can rewrite Eq.(A1) so that large x limit is easily taken,

$$\begin{aligned} \omega_{\lambda}^R(-x) &= -\frac{1}{\pi} \int_R d\zeta e^{xf(\zeta)}, \\ f(\zeta) &= i \sin \zeta - \sigma \zeta = u(\xi, \eta) + iv(\xi, \eta), \end{aligned} \quad (\text{A3})$$

where the real part u and imaginary part v of f are given respectively by,

$$\begin{aligned} u(\xi, \eta) &= -\cos \xi \sinh \eta - \sigma \xi, \\ v(\xi, \eta) &= \sin \xi \cosh \eta - \sigma \eta. \end{aligned} \quad (\text{A4})$$

We apply the steepest descent method and obtain the approximate form in the large x limit. We first find a saddle point of u .

$$\frac{\partial u}{\partial \xi} = 0, \quad \frac{\partial u}{\partial \eta} = 0. \quad (\text{A5})$$

The following conditions are satisfied at the saddle point.

$$\sin \xi \sinh \eta - \sigma = 0, -\cos \xi \cosh \eta = 0. \quad (\text{A6})$$

The saddle point for u which lies in the range $\xi \in [0, \pi]$ is

$$\zeta_{0R} = (\xi_0, \eta_0) = \left(\frac{\pi}{2}, \sinh^{-1} \sigma\right). \quad (\text{A7})$$

Along the curve which passes the saddle point ζ_{0R} with steepest descent, v is constant.

$$v(\xi, \eta) = v(\xi_0, \eta_0). \quad (\text{A8})$$

One can solve Eq.(A8) and obtain ξ as a function of η .

$$\xi_R(\eta) = \theta(\eta - \eta_0) \sin^{-1}\left(\frac{\cosh \eta_0 - \sigma(\eta - \eta_0)}{\cosh \eta}\right) + \theta(\eta_0 - \eta) \left[\pi - \sin^{-1}\left(\frac{\cosh \eta_0 - \sigma(\eta - \eta_0)}{\cosh \eta}\right)\right]. \quad (\text{A9})$$

The contour $\xi_R(\eta)$ is shown in Fig. 10 as the thick solid curve. One can rewrite the contour integration of the integral representation in Eq.(A1) as,

$$\omega_\lambda^R(-x) = -\frac{1}{\pi} e^{x(u(\xi_0, \eta_0) + iv(\xi_0, \eta_0))} \int_{-\infty}^{+\infty} d\eta \left(\frac{d\xi_R}{d\eta} + i\right) e^{x\{u(\xi_R(s), \eta) - u(\xi_0, \eta_0)\}}. \quad (\text{A10})$$

We carry out the integration with Gaussian approximation. One expands the real part of $f(\zeta)$ around at the saddle point.

$$u_R(\eta) \equiv u(\xi_R(\eta), \eta) = u(\xi_0, \eta_0) + \frac{1}{2!} \frac{d^2 u_R}{d\eta^2} \Big|_{\eta=\eta_0} (\eta - \eta_0)^2 + \dots \quad (\text{A11})$$

Truncating the series up to the term quadratic with respect to $\eta - \eta_0$ and replacing $\frac{d\xi_R}{d\eta}$ with $\frac{d\xi_R}{d\eta} \Big|_{\eta=\eta_0}$, we obtain

$$\begin{aligned} \omega_\lambda^R(-x) &\sim -\frac{1}{\pi} e^{x(u(\xi_0, \eta_0) + iv(\xi_0, \eta_0))} \left(\frac{d\xi_R}{d\eta} + i\right) \Big|_{\eta=\eta_0} \int_{-\infty}^{\infty} d\eta e^{\frac{x}{2} \frac{d^2 u_R}{d\eta^2} \Big|_{\eta=\eta_0} (\eta - \eta_0)^2}, \\ &= -\frac{1}{\pi} e^{xf(\zeta_0)} \left(\frac{d\xi_R}{d\eta} + i\right) \Big|_{\eta=\eta_0} \sqrt{\frac{2\pi}{-x \frac{d^2 u_R}{d\eta^2} \Big|_{\eta=\eta_0}}}. \end{aligned} \quad (\text{A12})$$

One finds,

$$\begin{aligned} \frac{d\xi_R}{d\eta} \Big|_{\eta=\eta_0} &= -1, \\ \frac{d^2 u_R}{d\eta^2} \Big|_{\eta=\eta_0} &= -2 \cosh \eta_0. \end{aligned} \quad (\text{A13})$$

Therefore, for small H limit, the Hankel function is given as,

$$\omega_\lambda^R(-x) = \sqrt{\frac{2}{\pi x}} \left(\frac{1}{1 + \sigma^2} \right)^{\frac{1}{4}} e^{-i\frac{\pi}{4} - \frac{\sigma x \pi}{2} + ix(\sqrt{1 + \sigma^2} - \sigma \sinh^{-1} \sigma)}. \quad (\text{A14})$$

Since we derive the approximate form of the Hankel functions, one can just substitute it into Eq.(76) and Eq.(77). We note the Eq.(77) is independent of the normalization of the solution. Therefore one can simply substitute

$$H_{i\rho_n}(\eta) \simeq \left(\frac{1}{a^2} + \sigma_n^2 \right)^{-\frac{1}{4}} e^{ixf(a, \sigma_n)}, \quad (\text{A15})$$

where $f(a, \sigma_n)$ is defined as,

$$f(a, \sigma_n) = \sqrt{\frac{1}{a^2} + \sigma_n^2} - \sigma_n \sinh^{-1} \sigma_n a. \quad (\text{A16})$$

One also obtains the derivative of $H_{i\rho_n}$,

$$H'_{i\rho_n}(\eta) \simeq ia \left(\frac{1}{a^2} + \sigma_n^2 \right)^{\frac{1}{4}} e^{ixf(a, \sigma_n)}. \quad (\text{A17})$$

Using the results, one obtains the functions in Eq.(76).

$$W_n[\eta, \eta_0] \simeq 2i \left(\frac{1 + \sigma_n^2}{\frac{1}{a^2} + \sigma_n^2} \right)^{\frac{1}{4}} \cos x(f(a, \sigma_n) - f(1, \sigma_n)), \quad (\text{A18})$$

$$Z_n[\eta, \eta_0] \simeq -2i \left((1 + \sigma_n^2) \left(\frac{1}{a^2} + \sigma_n^2 \right) \right)^{-\frac{1}{4}} \sin x(f(a, \sigma_n) - f(1, \sigma_n)), \quad (\text{A19})$$

$$W'_n[\eta, \eta_0] \simeq 2ia \left((1 + \sigma_n^2) \left(\frac{1}{a^2} + \sigma_n^2 \right) \right)^{\frac{1}{4}} \sin x(f(a, \sigma_n) - f(1, \sigma_n)), \quad (\text{A20})$$

$$Z'_n[\eta, \eta_0] \simeq 2ia \left(\frac{\frac{1}{a^2} + \sigma_n^2}{1 + \sigma_n^2} \right)^{\frac{1}{4}} \cos x(f(a, \sigma_n) - f(1, \sigma_n)). \quad (\text{A21})$$

One substitutes the approximate formulas for the functions given in Eq.(A18)-(A21) and obtains,

$$\begin{aligned} < j_0(x^0) > = e^{-3Hx^0} \int \frac{d^3k}{(2\pi)^3} \frac{\sinh \beta\mu}{(\cosh \beta\omega(k) - \cosh \beta\mu)} \\ & \frac{1}{2} \left[\left\{ \left(\frac{\omega_1 \omega_1(x^0)}{\omega_2 \omega_2(x^0)} \right)^{\frac{1}{2}} + \left(\frac{\omega_2 \omega_2(x^0)}{\omega_1 \omega_1(x^0)} \right)^{\frac{1}{2}} \right\} \sin x(f(a, \sigma_1) - f(1, \sigma_1)) \sin x(f(a, \sigma_2) - f(1, \sigma_2)) \right. \\ & \left. + \left\{ \left(\frac{\omega_1 \omega_2(x^0)}{\omega_2 \omega_1(x^0)} \right)^{\frac{1}{2}} + \left(\frac{\omega_2 \omega_1(x^0)}{\omega_1 \omega_2(x^0)} \right)^{\frac{1}{2}} \right\} \cos x(f(a, \sigma_1) - f(1, \sigma_1)) \cos x(f(a, \sigma_2) - f(1, \sigma_2)) \right], \end{aligned} \quad (\text{A22})$$

where $\omega_i(x^0)$ ($i = 1, 2$) are time dependent energies defined as,

$$\omega_i(x^0) = \sqrt{\frac{k^2}{a(x^0)^2} + m_i^2}, \quad (\text{A23})$$

while ω_i is independent of the time.

$$\omega_i = \omega_i(x^0 = 0) = \sqrt{k^2 + m_i^2}. \quad (\text{A24})$$

In the vanishing limit of H , one obtains,

$$x(f(a, \sigma_n) - f(1, \sigma_n)) \simeq -\omega_n x^0. \quad (\text{A25})$$

Therefore, in the limit, the current density is given as follows,

$$\begin{aligned} \langle j_0(x^0) \rangle = & \int \frac{d^3 k}{(2\pi)^3} \frac{\sinh \beta \mu}{(\cosh \beta \omega(k) - \cosh \beta \mu)} \\ & \left[\cos(\omega_1 - \omega_2)x^0 + \frac{1}{2} \left\{ \frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} - 2 \right\} \sin \omega_1 x^0 \sin \omega_2 x^0 \right]. \end{aligned} \quad (\text{A26})$$

Appendix B: The formula for the case $B = 0, \alpha_2 \rightarrow 0$

In this appendix, we give the outline of the derivation for the vanishing limit of the particle number violating mass term, i.e., $B \rightarrow 0$ and $\alpha_2 \rightarrow 0$. In this limit, two mass eigen values of the real scalars are degenerate and one can set $\bar{m}_1(x^0) = \bar{m}_2(x^0)$ in Eq.(48). Then one can readily show the following equation.

$$\frac{d}{dx^0} \log \langle j(x^0) \rangle = -3 \frac{d}{dx^0} \log a(x^0), \quad (\text{B1})$$

which leads to Eq.(80).

Acknowledgments

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